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SECOND ORDER DISPLACEMENT FIELDS OF A THICK ELASTIC PLATE UNDER THRUST—THE INCOMPRESSIBLE CASE

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Abstract—Branching of a thick incompressible plate of hyperelastic material under uniaxial thrust is studied. The acceptable barreling second order displacement field is described, corresponding to the first order lower flexure mode. An application, clarifying the theory, is presented for a Mooney–Rivlin material.

1. INTRODUCTION

Following the theory of superposition of infinitesimal deformations on finite deformations of elastic bodies (Green *et al.*, 1952), an extensive literature concerning stability problems of thick elastic bodies has evolved (Wilkes, 1955; Fosdick and Shield, 1963; Biot, 1965; Kerr and Tang, 1967 Levinson, 1968; Nowinski, 1969; Wu and Widera, 1969; Sawyers and Rivlin, 1974, 1982). Using this theory, the critical conditions (loading) could be located along with the definition of the eigendeformations.

Similar problems from the strength of materials point of view are discussed in the classical textbook of Timoshenko *et al.* (1963) concerning elastic stability theory. However, for simple strength of materials problems, post-stability studies have been performed using theories initiated by Koiter (1945) and Thompson *et al.* (1973).

These theories have been supported by the mathematical branching analysis of non-linear equations (Vaingberg and Trenogin, 1974; Chow and Hale, 1982). Furthermore, catastrophe theory (Thom, 1975), greatly helped in understanding the contribution of imperfections in the stability analysis of structures.

Although the theory of small deformations superimposed on large ones (Green *et al.*, 1952) yields a first estimate of the critical conditions, it is inadequate for post-stability studies. By post-stability analysis we mean the definition of the equilibrium paths in the post-critical regions and the study of their stability character. Consequently, post-stability analysis requires higher than first order terms of the displacement vector around the critical placement, whereas the aforementioned theory deals only with the first order terms.

On the other hand, an extension of Signorini's expansion (Truesdell and Noll, 1965; Wang and Truesdell, 1973) when the reference placement is stressed (Capriz and Guidugli, (1979) may be a suitable analytical method for studying those topics. In fact Truesdell and Noll (1965) derived branching conditions based on Signorini's expansion with stressed reference placement. It is pointed out that not only Signorini, but also Truesdell and Noll (1965), and Wang and Truesdell (1973), who revised the method, were very cautious in using the expansion in stressed reference placements. It is the author's belief that the method generally fails in branching problems. Nevertheless, Signorini's expansion could be applied in stressed reference placements in unique equilibrium positions. Improperly, the method may be applied in branching problems when the kernel (space of eigenfunctions) is one-dimensional. For multidimensional kernels of the branching problems Signorini's expansion fails. Although the topic is quite important and is going to be presented soon in other work, Signorini's method in the context of branching analysis is outlined in the first chapter.

The motive for the present study was Sawyers and Rivlin's (1982) work, where formal stability analysis is attempted on a thick elastic plate under uniaxial thrust, using Koiter's post-buckling theory. Such an analysis requires higher order terms of the displacement vector. The present work might be considered as a first step in that direction. It should be pointed out that no problem concerning thick bodies, with nonhomogeneous deformations, with a complete stability study is known. This is due to the lengthy underlying algebra. The recently developed computerized algebra software, such as Mathematica, has become an indispensable tool for performing such an analysis.

In general terms the present study considers a hyperelastic plate under uniaxial thrust. The second order displacement field is described for the antisymmetric mode (Sawyers and Rivlin, 1974). It is worth mentioning that the main difference between the first order displacement field and the second order one lies in the type of linearized equilibrium systems. Indeed, for the first order displacement field the equilibrium system is homogeneous while for the second it is affine, since the first order displacement field induces the non-homogeneous terms of the affine equilibrium system. The non-homogeneous part of the governing system needs some specific treatment discussed in Chapter 6, but is mainly clarified in the application.

Furthermore, the boundary conditions on the forced boundary have been changed. Since the problem described by Sawyers and Rivlin (1974), leads to a unilateral contact second order problem (Panagiotopoulos, 1985) whose closed form solution is almost impossible, beam-like boundary conditions have been adapted. These conditions have already been imposed in other problems of continuum mechanics, like the torsion (Rivlin, 1949) and the beam (Rivlin, 1948) problems.

As is evident, the second order displacement field depends on the first order one. Since the complete definition of the first requires the application of the Fredholm Alternative theorem (Vaingberg and Trenogin, 1974; Capriz and Guidugli, 1979), on the third order problem, the second order displacement field also is not completely defined. It is in fact presented as a mono-parametric displacement field directly dependent on the parameter of the first order solution.

The purpose of the present work is not only the discussion of the symmetric second order effects, but also the exposition of a method for the definition of higher order terms of the bifurcation equilibrium paths necessary for the complete stability study of the problem. Similar methods could be employed for the third or higher order problems.

Apart from some differences on the boundary conditions, the main setting of the first order problem will be borrowed from the work of Sawyers and Rivlin (1974); since only its results will be referred to this reference is indispensable for further reading of the present work.

Clearly, barreling second order displacement fields have been revealed accompanying the first order lower flexure mode. The existence of those barreling deformations have been discovered experimentally by Beatty and co-workers (1968, 1976). In fact the coexistence of both modes has been attributed to some kind of transition from the flexure to the barreling mode. In my opinion the coexistence of both modes is due to the coexistence of the first and second order displacement fields. This point should be scrutinized experimentally on the basis of the present study.

2. SUCCESSIVE APPROXIMATIONS OF THE TRACTION PROBLEM

Let a continuous body occupy a regular region B_0 of a three-dimensional euclidean space. This placement is stressed by a distributed traction $\boldsymbol{\tau}_0$ applied on the boundary ∂B_0 with outer unit normal n_0 and the distributed body forces $\mathbf{b}_0$ in B_0. Considering the current placement B loaded by the traction $\boldsymbol{\tau}$ and the body forces $\mathbf{b}$ in the neighborhood of B_0, we are looking for a Taylor expansion of the displacement field $\mathbf{u}$ around the reference placement B_0. Specifically, $\mathbf{u} = (u_1, u_2, u_3)$ is the displacement of a point from the stressed reference placement B_0 to the current one B, with the displacement gradient,

$$\mathbf{H} = \nabla \mathbf{u} \tag{1}$$

and the relative deformation gradient

$$\mathbf{F} = \mathbf{1}+\mathbf{H}. \tag{2}$$

Assuming a hyperelastic material, the constitutive equation for the Piola–Kirchhoff stress tensor $\mathbf{S}$ is expressed by:

$$\mathbf{S} = \Phi(\mathbf{F}). \tag{3}$$

The equilibrium equations for the traction problem are derived by the system:

$$\begin{aligned} -\operatorname{div}\Phi(\mathbf{F}) &= \mathbf{b}(\varepsilon) \quad \text{in } B_o \\ \Phi(\mathbf{F})n_o &= \boldsymbol{\sigma}(\varepsilon) \quad \text{on } \partial B_o, \end{aligned} \tag{4}$$

where $\mathbf{b}(\varepsilon)$ is the distributed load per unit mass and $\boldsymbol{\sigma}(\varepsilon)$ is the traction exerted on the boundary which depends on the small parameter ε and referred to the reference placement B_o. Likewise, recalling that the reference placement is an equilibrium placement of the elastic body,

$$\begin{aligned} -\operatorname{div}\Phi(\mathbf{1}) &= \mathbf{b}_o \quad \text{in } B_o \\ \Phi(\mathbf{1})\boldsymbol{n}_o &= \boldsymbol{\sigma}_o \quad \text{on } \partial B_o, \end{aligned} \tag{5}$$

where the relative deformation gradient $\mathbf{F}$ becomes $\mathbf{1}$ at the reference placement loaded by the device,

$$l_o = (\mathbf{b}_o, \boldsymbol{\sigma}_o). \tag{6}$$

As we are looking for Taylor expansion of the displacement field around the stressed reference placement B_o, the governing equation (4) is expanded around the reference placement and gives:

$$\begin{aligned} -\operatorname{div}\nabla\Phi\,|_{\mathbf{1}}(\mathbf{H}) &= \mathbf{b}(\varepsilon) - \boldsymbol{b}_o + \operatorname{div}\mathbf{G}(\mathbf{H}) = \boldsymbol{B} \\ \nabla\Phi\,|_{\mathbf{1}}(\mathbf{H})\cdot\boldsymbol{n}_o &= \boldsymbol{\sigma}(\varepsilon) - \boldsymbol{\sigma}_o - \mathbf{G}(\mathbf{H})\cdot\boldsymbol{n}_o = \boldsymbol{T}, \end{aligned} \tag{7}$$

where $\mathbf{G}(\mathbf{H})$ is the nonlinear part of the expansion $\Phi(\mathbf{H})$ around $\mathbf{H} = \mathbf{0}$. It is mentioned, though apparent, that the L.H.S. of eqns (7) are linear, whereas the R.H.S. are non-linear.

The key point is the existence or not of the kernel of the linear part of the system (7). In fact if the system

$$\begin{aligned} -\operatorname{div}\nabla\Phi\,|_{\mathbf{1}}(\nabla(\mathbf{v})) &= \mathbf{0} \\ \nabla\Phi\,|_{\mathbf{1}}(\nabla(\mathbf{v}))\cdot\boldsymbol{n}_o &= \mathbf{0} \end{aligned} \tag{8}$$

accepts only the trivial (zero) solution, no kernel of the linear problem exists, the non-linear problem accepts an unique solution and Signorini's expansion could be found here by splitting the nonlinear problem in linearized problems of the same order of magnitude of the parameter ε as in Capriz and Guidugli (1979). Hence in this specific case the extension of Signorini's expansion from the unstressed reference placement to the stressed one is feasible.

Indeed the expansions

$$\mathbf{u} = \varepsilon\dot{\mathbf{u}} + \varepsilon^2\ddot{\mathbf{u}} + \cdots + \varepsilon^n \overset{(n)}{\mathbf{u}}$$

$$\boldsymbol{B} = \varepsilon\dot{\boldsymbol{B}} + \varepsilon^2\ddot{\boldsymbol{B}} + \cdots + \varepsilon^n \overset{(n)}{\boldsymbol{B}}$$

$$\boldsymbol{T} = \varepsilon\dot{\boldsymbol{T}} + \varepsilon^2\ddot{\boldsymbol{T}} + \cdots + \varepsilon^n \overset{(n)}{\boldsymbol{T}} \tag{9}$$

are valid and $\overset{(n)}{\mathbf{u}}$ is found as a solution of the linearized system,

$$-\operatorname{div} \nabla\Phi\,|_1(\nabla\,\overset{(n)}{(\mathbf{u})}) = \overset{(n)}{\boldsymbol{B}}$$

$$\nabla\Phi\,|_1(\nabla\,\overset{(n)}{(\mathbf{u})})\cdot \boldsymbol{n}_o = \overset{(n)}{\boldsymbol{T}}, \tag{10}$$

where the nonlinear terms (R.H.S.) of the system (10) depend on the loading device up to the n^{th} order and the terms of $\mathbf{u}$ which are of order at most equal to $n-1$. Thus $\overset{(n)}{\mathbf{u}}$ appears only in the linear terms of the system (10) and at each step $\overset{(n)}{\boldsymbol{B}}$ and $\overset{(n)}{\boldsymbol{T}}$ are completely defined by the lower order terms $\dot{\mathbf{u}}, \ddot{\mathbf{u}}, \ldots, \overset{(n-1)}{\mathbf{u}}$ of the displacement field.

However, when the kernel of the linear system exists, i.e. nontrivial solutions $\mathbf{v}_i$ exist and non-unique solutions of the nonlinear system (7) are expected, branching of the equilibrium solutions takes place with

$$\mathbf{u} = \sum_{i=1}^{k} \xi_i \mathbf{v}_i + 0\Big(\sum_{i=1}^{k} |\xi_i|^2\Big), \tag{11}$$

where $\mathbf{v}_i$ is a base of the kernel, ξ_i are small parameters that should be defined by the Fredholm Alternative theorem (Vaingberg and Trenogin, 1974; Capriz and Guidugli, 1979) which is the main tool of the branching theory of continuous systems. In reality, on using that theorem a continuous system reduces to an algebraic one.

Put simply, the Fredholm Alternative theorem requires for the existence of a small solution of the nonlinear system the condition (Capriz and Guidugli, 1979),

$$\langle \boldsymbol{l}_n, \mathbf{v}_i \rangle = \int_{B_o} \overset{(n)}{\boldsymbol{B}}\, \mathbf{v}_i \, \mathrm{d(vol)} + \int_{\partial B_o} \overset{(n)}{\boldsymbol{T}}\, \mathbf{v}_i \, \mathrm{d(surf)} = 0 \tag{12}$$

Using this relation the equilibrium path is completely defined.

However, when the kernel of the linear system (8) exists, i.e. nontrivial solutions $\mathbf{v}_i$ exist and eqns (12) describe the branching of the equilibrium paths through the definition of the parameters ξ_i, singularity theory should be applied on that system (Lazopoulos, 1994).

It is evident that ξ_i might not be of the same order of magnitude as ε, neither should the various ξ_i be of the same order. In reality, the orders of the ξ_i are fractional with respect to the small parameter ε. Consequently, in the present case of branching, Signorini's expansion fails, since we do not expect linearization of the nonlinear system (7).

However, when the kernel is one-dimensional some adjustments between ε and ξ_1 could be devised so that the problem might have the structure of Signorini's expansion. The problem of the uniaxial thrust on an elastic plate is included in this case. In reality, it is more convenient to use branching theory in Signorini's expansion, if possible, than immediately performing branching theory without the expansion.

In the specific case of unidimensional branching, linearization on the governing system may be performed and systems similar to (10) are shown. Likewise, the Fredholm Alternative condition requiring the existence of the solutions $\overset{(n)}{\mathbf{u}}$ is expressed by eqn (12). To be more specific, the procedure works as follows. At each n^{th} step the solution of the linearized system (10) is defined by:

$$\overset{(n)}{\mathbf{u}} = \xi_n \mathbf{v} + \overset{(n)}{\mathbf{v}}{}^{\mathrm{p}}_n + 0(\varepsilon^n), \tag{13}$$

where $\mathbf{v}_n^{\mathrm{p}}$ is the particular solution of the system that depends on $\overset{(n)}{\boldsymbol{B}}$ and $\overset{(n)}{\boldsymbol{T}}$ including terms of $\dot{\mathbf{u}}, \ddot{\mathbf{u}}, \ldots, \overset{(n-1)}{\mathbf{u}}$. The solution is defined apart from the coefficient ξ_n. That coefficient is defined by the Fredholm Alternative condition of the next $(n+1)^{\text{th}}$ order system. In fact the nonhomogeneous parts (R.H.S.) of the linearized system (10) ($\overset{(n)}{\boldsymbol{B}}$, $\overset{(n)}{\boldsymbol{T}}$) include terms of the displacement $\mathbf{u}$ of order less than or equal to $n-1$. Consequently, similar terms for the $(n+1)^{\text{th}}$ order linearized system include terms of $\mathbf{u}$ up to the nth order. Therefore, applying the Fredholm Alternative condition to the $(n+1)^{\text{th}}$ linearized system the unique unknown is the coefficient ξ_n. The solutions of ξ_n define the equilibrium paths of the system. This general theory is adjusted to the specific problem under study.

Finally, the complete study of the traction problem requires the consideration of the zero total moment of momentum, a topic much discussed (Truesdell and Noll, 1965; Wang and Truesdell, 1973). Since that point is not important for the specific problem of the plate under discussion, it is not considered further.

3. THE PLANE STRAIN EQUILIBRIUM PROBLEM OF A THICK INCOMPRESSIBLE PLATE

Consider an incompressible elastic plate of dimensions $2l_1$, $2l_2$, and l_3 almost infinite. A coordinate system (X_1, X_2, X_3) is parallel to the edges of the plate (see Fig. 1). Furthermore, the thrust is applied at the surfaces $X_1 = \pm l_1$ only and the deformation takes place in the plane (X_1, X_2). The points $(X_1, X_2) = (\pm l_1, 0)$ are constrained to move on the line $(X_1, 0)$. These supports undertake shear on the forced surfaces $X_1 = \pm l_1$ owing to friction.

If we try to be consistent, not only with the first order problem, but also with the second order one, beam-like boundary conditions should be adapted, i.e. the resultant force applied on the surfaces $X_1 = \pm l_1$ should be equal to T, whereas the resultant moment should be zero.

Assuming that the thick elastic body is subjected to a plane deformation, every material point (X_1, X_2) is displaced to the current placement (x_1, x_2) lying in the same plane. Then

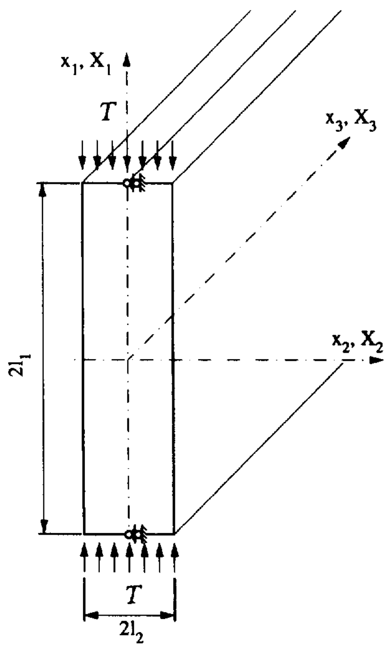

Fig. 1. The geometry of the thick plate.

the deformation gradient (Truesdell and Noll, 1965, Wang and Truesdell, 1973) is defined by:

$$\mathbf{F} = \left[\frac{\partial x_i}{\partial X_j}\right] \quad i,j = 1,2,3. \tag{14}$$

Furthermore the left Caushy–Green deformation tensor **B** is assigned by:

$$\mathbf{B} = \mathbf{F}^*\mathbf{F}^{\mathrm{T}}, \tag{15}$$

where $(^{\mathrm{T}})$ denotes the transpose tensor. Likewise the principal strain invariants are defined by:

$$I_1 = \mathbf{B}_{kk} = \mathrm{tr}\mathbf{B}$$

$$I_2 = 0.5^*(\mathbf{B}_{kk}\mathbf{B}_{LL} - \mathbf{B}_{kL}\mathbf{B}_{kL}) = 0.5^*(\mathrm{tr}\mathbf{B})^2 - 0.5^*\,\mathrm{tr}(\mathbf{B}^2). \tag{16}$$

Because of the incompressibility condition,

$$I_3 = \det(\mathbf{B}) = 1. \tag{17}$$

The strain energy density is expressed by the function:

$$W = W(I_1, I_2)$$

with $W(3,3) = 0$. Likewise the first Piola–Kirchhoff stress tensor is defined by (Truesdell and Noll, 1965; Wang and Truesdell, 1973):

$$\boldsymbol{S} = \boldsymbol{\sigma} = -P\mathbf{F}^{-\mathrm{T}} + 2(W_1 + I_1 W_2)\mathbf{B}\mathbf{F}^{-\mathrm{T}} - 2W_2(\mathbf{B}\cdot\mathbf{B})\mathbf{F}^{-\mathrm{T}}, \tag{18}$$

where P is an arbitrary hydrostatic pressure imposed by the incompressibility constraint, eqn (17). The sub-indices of W indicate differentiation with respect to the principal invariants, i.e.

$$W_i = \frac{\partial W(I_1, I_2)}{\partial I_i}, \quad W_{ij} = \frac{\partial W(I_1, I_2)}{\partial I_i \partial I_j}.$$

The equilibrium equations are defined by:

$$\frac{\partial \sigma_{ij}}{\partial X_i} = 0 \quad \text{in}\,\Omega, \tag{19}$$

the beam-like boundary conditions:

$$\mathrm{T} = \int_{-l_2}^{l_2} \sigma_{11}\,\mathrm{d}X_2, \quad \int_{-l_2}^{l_2} \sigma_{11} X_2\,\mathrm{d}X_2 = 0 \quad \text{on } X_1 = \mp l_1$$

$$\sigma_{21} = \sigma_{22} = 0 \quad \text{on } X_2 = \mp l_2 \tag{20}$$

and the axis of the plane is simply supported (see Fig. 1). This condition has some physical sense, since friction undertakes the shear of the forced surfaces of the plate.

Let us consider a finite homogeneous deformation defined by the principal stretches $(\lambda_1, \lambda_2, 1)$ (Truesdell and Noll, 1965; Wang and Truesdell, 1973) engendered by the force T. Perturbing the force T so that

$$\mathrm{T} = \mathring{\mathrm{T}} + \varepsilon^2\ddot{\mathrm{T}} + 0(\varepsilon^3), \tag{21}$$

where $0(\varepsilon^3)$ denotes the terms of the same or higher order of ε^3, the resulting incremental displacement vector $\mathbf{u}$ is assumed to be expressed by the series expansion:

$$\mathbf{u} = \varepsilon\dot{\mathbf{u}} + \varepsilon^2\ddot{\mathbf{u}} + 0(\varepsilon^3). \tag{22}$$

Hence the deformation gradient $\mathbf{F}$, eqn (14), equals:

$$\mathrm{F}_{ij} = \lambda_i\delta_{ij} + u_{i,j}\ \text{for}\ i,j = 1,2 \tag{23}$$

$\mathrm{F}_{ij} = \delta_{ij}$ for the other tensor elements, where the comma denotes differentiation with respect to $X_j, \ldots, j = 1, 2$. Finally the series expansion for the hydrostatic pressure P is given by:

$$P = \mathring{P} + \varepsilon\dot{P} + \varepsilon^2\ddot{P} + 0(\varepsilon^3), \tag{24}$$

whereas the stress tensor σ is expressed by:

$$\boldsymbol{\sigma} = \mathring{\boldsymbol{\sigma}} + \varepsilon\dot{\boldsymbol{\sigma}} + \varepsilon^2\ddot{\boldsymbol{\sigma}} + 0(\varepsilon^3) \tag{25}$$

Recalling the traction and displacement perturbation expansions, eqns (21) and (22), and the expression for the Piola–Kirchhoff stress tensor, eqn (25), the equilibrium equations break into the following linearized systems:

(a) The first order equilibrium system:

$$\operatorname{div}(\dot{\boldsymbol{\sigma}}) = \operatorname{div}(\Sigma\dot{\mathbf{u}}) = 0, \quad \Sigma \text{ a linear operator},$$

$$\int_{-l_{-2}}^{l_2} \dot{\sigma}_{11}\,\mathrm{d}X_2 = 0 \quad \int_{-l_2}^{l_2} \dot{\sigma}_{11}X_2\,\mathrm{d}X_2 = 0, \quad \text{at} \quad X_1 = \mp l_1$$

$$\dot{\sigma}_{22} = \dot{\sigma}_{21} = 0 \quad \text{at} \quad X_2 = \pm l_2$$

$$\dot{u}_2(\mp l_1, 0) = 0. \tag{26}$$

(b) The second order equilibrium system:

$$-\operatorname{div}(\Sigma\ddot{\mathbf{u}}) = \ddot{B}(\dot{\mathbf{u}}, \ddot{T}) \quad \text{in}\,\Omega$$

$$\ddot{T} = \int_{-l_2}^{l_2} \ddot{\sigma}_{11}\,\mathrm{d}X_2\,, \quad \int_{-l_2}^{l_2} \ddot{\sigma}_{11}X_2\,\mathrm{d}X_2 = 0\,, \quad \text{at} \quad X_1 = \mp l_1$$

$$\ddot{\sigma}_{22} = \ddot{\sigma}_{21} = 0\,, \quad \text{at} \quad X_2 = \mp l_2$$

$$\ddot{u}_2(\mp l_1, 0) = 0, \tag{27}$$

where $\ddot{\mathbf{B}}(\dot{u}, \ddot{\mathbf{T}})$ is the nonlinear part of the equation that depends on the first order displacement field $\dot{\mathbf{u}}$ and the incremental thrust parameter $\ddot{\mathrm{T}}$. The boundary conditions of the present second order problem conform with beam-like boundary conditions, eqn (20). Likewise, the incompressibility condition requires the third strain invariant to be equal to one. Hence the higher order terms of the third invariant satisfy the following equations:

$$\lambda_1\dot{u}_{2,2} + \lambda_2\dot{u}_{1,1} = 0$$

$$\lambda_1\ddot{u}_{2,2} + \lambda_2\ddot{u}_{1,1} + \dot{u}_{1,1}\dot{u}_{2,2} + \dot{u}_{1,2}\dot{u}_{2,1} = 0. \tag{28}$$

The initial (pre-buckling) homogeneous placement with stretches (λ_1, λ_2, 1) is described by the equations:

$$\mathring{T} = 2l_2(\mathring{\sigma}_{11}) \tag{29a}$$

$$\mathring{\sigma}_{22} = \mathring{\sigma}_{21} = 0 \tag{29b}$$

$$\lambda_1\lambda_2 = 1. \tag{29c}$$

Likewise eqn (29b) defines the constant pressure $\mathring{P}$, see eqn (3.5) of Sawyers and Rivlin (1974) by:

$$\mathring{P} = 2\lambda_2^2[W_1+(1+\lambda_1^2)W_2]. \tag{30}$$

The constant stretches λ_1, λ_2 are defined by eqn (17c) and

$$\mathring{\sigma}_{11} = \mathring{T}/(2l_2) = (2/\lambda_1)(\lambda_1^2-\lambda_2^2)(W_1+W_2), \tag{31}$$

see eqn (3.6) of Sawyers and Rivlin (1974).

4. THE FIRST ORDER DISPLACEMENT FIELD

Following Sawyers and Rivlin (1974), the first order equilibrium system (26a) with the incompressibility condition (28a) is expressed by the system:

$$(1+\gamma_1)\dot{u}_{1,11}+\dot{u}_{1,22}-\lambda_2\dot{p}_{,1} = 0 \tag{32}$$

$$\dot{u}_{2,11}+(1+\gamma_2)\dot{u}_{2,22}-\lambda_1\dot{p}_{,2} = 0\lambda_2\dot{u}_{1,1}+\lambda_1\dot{u}_{2,2} = 0 \tag{33}$$

where

$$\gamma_1 = 2(\lambda_1^2-\lambda_2^2)(W_{11}+(\lambda_2^2+2)W_{12}+(1+\lambda_2^2)W_{22})/(W_1+W_2)$$
$$\gamma_2 = 2(\lambda_2^2-\lambda_1^2)(W_{11}+(\lambda_1^2+2)W_{12}+(1+\lambda_1^2)W_{22})/(W_1+W_2)$$
$$\dot{p} = 0.5P/(W_1+W_2). \tag{34}$$

The incremental problem has to satisfy the boundary conditions:

$$\dot{\sigma}_{21} = 2(W_1+W_2)(u_{1,2}+\lambda u_{2,1}) = 0 \quad \text{on } X_2 = \pm l_2 \tag{35a}$$

$$\dot{\sigma}_{22} = 2(W_1+W_2)[(2+a_2)u_{2,2}-\lambda_1\dot{p}] = 0 \text{ on } X_2 = \pm l_2 \tag{35b}$$

$$\int_{-l_2}^{l_2} \dot{\sigma}_{11}\,dX_2 = 0 \quad \int_{-l_2}^{l_2} \dot{\sigma}_{11}X_2\,dX_2 = 0, \quad \text{at } X_1 = \mp l_1 \tag{35c}$$

with

$$\dot{\sigma}_{11} = 2(W_1+W_2)[(2+a_1)u_{1,1}-\lambda_2\dot{p}] = 0 \quad \text{on } X_1 = \pm l_1$$
$$\dot{u}_2(\mp l_1, 0) = 0 \tag{35d}$$

with

$$\lambda = \lambda_2/\lambda_1.$$

Furthermore, the homogeneous problem of eqns (32)–(35) accepts the flexure solution:

$$\dot{u}_1 = -\xi_1 \sin(\Omega X_1) U_1(X_2) \tag{36a}$$

$$\dot{u}_2 = \xi_1 \cos(\Omega X_1) U_2(X_2) \tag{36b}$$

$$\dot{p} = \xi_1 \cos(\Omega X_1) P(X_2), \tag{36c}$$

where

$$\Omega = \pi/2$$

$$U_2(X_2) = M\cosh(\Omega_1 X_2) + \cosh(\Omega_2 X_2) \tag{36d}$$

and

$$U_1(X_2) = (\lambda\Omega)^{-1}\{\Omega_1 M \sinh(\Omega_1 X_2) + \Omega_2 \sinh(\Omega_2 X_2)\} \tag{36e}$$

$$P(X_2) = (\lambda_2\lambda\Omega)^{-1}\{\Omega_1(\Omega_1^2 - (1+\gamma_1)\Omega^2) M \sinh(\Omega_1 X_2) + \Omega_2(\Omega_2^2 - (1+\gamma_1)\Omega^2)\sinh(\Omega_2 X_2)\} \tag{36f}$$

with

$$\Omega_1^2, \Omega_2^2 = 0.5*\Omega^2[1+\lambda^2 + A(1-\lambda)^2 \pm [\{1+\lambda^2+A(1-\lambda)^2\}^2 - 4\lambda^2]^{1/2}] \tag{36g}$$

provided $\Omega_1^2 \neq \Omega_2^2$ with

$$A = 2(\lambda_1+\lambda_2)^2(W_{11}+2W_{12}+W_{22})/(W_1+W_2)$$

and

$$M = -\frac{\Omega_2^2 + \lambda^2\Omega^2}{\Omega_1^2+\lambda^2\Omega^2}\frac{\cosh(\Omega_2 l_2)}{\cosh(\Omega_1 l_2)}, \tag{37a}$$

where

$$\frac{\tanh(\Omega_2 l_2)}{\tanh(\Omega_1 l_2)} = \left[\frac{\Omega_2+\lambda\Omega}{\Omega_1^2+\lambda^2\Omega^2}\right]^2 \frac{\Omega_1}{\Omega_2}. \tag{37b}$$

Some questions might be raised here about the differences of the present kernel and the kernel exhibited in Sawyers and Rivlin (1974). Indeed the present homogeneous problem seems to have relaxed boundary conditions with respect to the one in Sawyers and Rivlin (1974). Therefore, a larger kernel should be expected here. It turns out that this is not the case. Apart from some minor difference, the kernels are quite similar.

In fact, the major difference is located in the Ω value. In the present case the lower mode corresponds to $\Omega = 0.5\,\pi/l_1$, while in the case presented in Sawyers and Rivlin (1974) $\Omega = \pi/l_1$. But despite that difference both kernels are quite similar.

Indeed conditions (36)–(37) are the same as derived by the governing equations (32)–(33) and the boundary conditions (35) on the boundaries $X_2 = \pm l_2$. These conditions are the same in both studies. The question raised concerns the existence of a larger kernel in the present case since, comparatively, the boundary conditions at $X_1 = \pm l_1$ have been relaxed.

Trying to locate solutions of the homogeneous system with the lower mode defined by $\Omega \neq 0.5\ \pi/l_1$, then although the boundary conditions (35a,b) on $X_2 = \pm l_2$ are valid, the boundary conditions on $X_1 = \pm l_1$ require, at $X_1 = \pm l_1$, the boundary conditions (35c) standing for zero first order thrust and zero first order total (bending) moment. Likewise, the constraint expressed by eqn (35d) has to be satisfied.

Since $u_{1,1}$ and P are anti-symmetric with respect to X_2, the first of the boundary conditions (35c) is identically satisfied. Likewise, the condition (35d) evaluates with $(\Omega \neq 0.5\pi/x_1)$ the coefficient M. Indeed,

$$M = -1. \tag{38}$$

Yet, eqns (37a,b) and (38) reveal that:

$$\Omega_2 \sinh(2\Omega_2 l_2) = \Omega_1 \sinh(2\Omega_1 l_1). \tag{39}$$

Recalling that the function

$$\Psi = X \sinh(2Xl_2)$$

is strictly increasing, eqn (39) is valid only when,

$$\Omega_2 = \Omega_1.$$

First, the case with $\Omega \neq 0.5\ \pi/l_1$ has been reduced to the specific case with $\Omega_1 = \Omega_2$ discussed in Section 6 of Sawyers and Rivlin (1974). Additionally, the second condition of eqn (35c), concerning the zero total moment, has to be satisfied. The author feels that further discussion on this specific point is quite marginal for the main subject, rejecting any solutions of the homogeneous problem eqns (32)–(35) with $\Omega \neq 0.5\ \pi/l_1$.

In conclusion, apart from the evident difference in the values of Ω, the kernels in the present work and the study presented by Sawyers and Rivlin (1974) are the same.

Flexure behavior is discussed because it is the first exhibited mode from the initial placement. It should be recalled that the present problem is the eigenvalue problem that defines the critical conditions (critical λ) and the ξ_1-parametric first order displacement field $\dot{\mathbf{u}}$.

4. THE SECOND ORDER DISPLACEMENT FIELD

The second order equilibrium system is given by eqn (27) and the corresponding incompressibility condition, eqn (28b). In fact, the governing equilibrium equations for the second order displacement field are expressed by:

$$(1+\gamma_1)\ddot{u}_{1,11} + \ddot{u}_{1,22} - \lambda_2 \ddot{p}_{,1} = B_1(\dot{u}_1, \dot{u}_2) \tag{40a}$$

$$\ddot{u}_{2,11} + (1+\gamma_2)\ddot{u}_{2,22} - \lambda_1 \ddot{p}_{,2} = B_2(\dot{u}_1, \dot{u}_2) \tag{40b}$$

$$\lambda_2 \ddot{u}_{1,1} + \lambda_1 \ddot{u}_{2,2} = B_3(\dot{u}_1, \dot{u}_2), \tag{40c}$$

whereas γ_1 and γ_2 have already been defined, eqns (34), and $\ddot{p} = 0.5\ \ddot{P}(W_1 + W_2)$. Yet the second order boundary conditions have to be satisfied, i.e.

$$\ddot{\sigma}_{22} = \ddot{\sigma}_{21} = 0 \quad \text{at } X_2 = \pm \mathbf{l}_2 \quad \text{and} \tag{40d}$$

$$\int_{-l_2}^{l_2} \ddot{\sigma}_{11}\,\mathrm{d}X_2 = \ddot{\mathrm{T}}, \quad \int_{-l_2}^{l_2} \ddot{\sigma}_{11} X_2\,\mathrm{d}X_2 = 0, \quad \text{at } X_1 = \pm l_2 \tag{40e}$$

$$\ddot{u}_2(l_1, 0) = 0. \tag{40f}$$

The system (40) has the form:

$$L(\ddot{\mathbf{u}}) = \ddot{B}(\dot{\mathbf{u}}, \ddot{\mathrm{T}}), \tag{41}$$

where L stands for the linear part of the system, whereas $\ddot{B}$ is the nonlinear operator dependent upon the first order displacement vector $\dot{\mathbf{u}}$, and the second order forcing $\ddot{\mathrm{T}}$. The system corresponds to eqn (10) with $n = 2$.

As the kernel of the linear system,

$$L(\mathbf{v}) = 0,$$

accepts the solution (36), according to branching theory,

$$\ddot{\mathbf{u}} = \xi_2 \mathbf{v} + \mathbf{u}^{\mathrm{p}}, \tag{42}$$

where $\mathbf{u}^{\mathrm{p}}$ is a particular solution of the system (39). Furthermore, owing to the nontrivial kernel, the existence of the small solution requires the Fredholm Alternative condition, eqn (14) with $n = 2$,

$$\int_{B_0}\int (\mathbf{v}_1 \ddot{B}_1 + \mathbf{v}_2 \ddot{B}_2)\,\mathrm{d}A + 2\ddot{T}_{|(1,X_2)} v_{,}(1,0) = 0, \tag{43}$$

with

$$\ddot{T}_{|(1,X_2)} = \ddot{T} - \int_{-l_2}^{l_2} (\ddot{\sigma}_{11} - 2(W_1 + W_2)[(2+a_1)\ddot{u}_{1,1} - \lambda_2 \ddot{p}])\,\mathrm{d}X_2.$$

It turns out that $\ddot{B}_1$ and $\ddot{B}_2$ are symmetric with respect to the X_1 axis, while $\mathbf{v}_1$, $\mathbf{v}_2$ are antisymmetric and

$$u_1(1,0) = \dot{u}_1(1,0) = 0. \tag{44}$$

Consequently, the Fredholm condition (43) is identically satisfied. On the other hand, we expected the evaluation of ξ_1 from the condition (43), see eqns (36), since $\ddot{\boldsymbol{B}}_1$ and $\ddot{\boldsymbol{B}}_2$ depend on the unique parameter ξ_1. However, the reduction of of the solvability condition to an identity does not allow the evaluation of the parameter ξ_1. Hence, we expect the definition of ξ_1 from the third order solvability condition, eqn (14) with $n = 3$.

In reality, the situation becomes more complicated now, because on the R.H.S. (nonlinear) part of the governing linearized equilibrium system, eqn (10), of the third order $n = 3$, the terms depend on $\dot{\mathbf{u}}$ and $\ddot{\mathbf{u}}$. Consequently, they include both parameters ξ_1 and ξ_2 and Fredholm condition alone is inadequate for the definition of the equilibrium paths.

However, help is offered by the symmetries of the problem. In fact, it should be symmetric with respect to the X_1 axis for opposite values of ξ_1. That is owing to the expected symmetric branching. In this case, ξ_2 should remain the same (since it is of the order ξ_1^2) for both cases ($\xi_1 > 0$ and $\xi_1 < 0$). However, the flexure kernel of the linear problem does not allow that kind of symmetry and the only acceptable solution for the second order displacement $\ddot{\mathbf{u}}$, eqn (42), that conforms with the symmetries is $\xi_2 = 0$. Hence the second order solution should be defined only by the particular solution, i.e.

$$\ddot{\mathbf{u}} = \mathbf{u}^{\mathrm{p}}.$$

The next step is the location of a partial solution $\mathbf{u}^{\mathrm{p}}$ of the affine system (38). This solution is feasible because following some elimination procedure a differential equation could be shown dealing only with the $\ddot{u}_1$ unknown variable, or with the $\ddot{u}_2$ one. Having found the u_2^{p} unknown and recalling the incompressibility condition, eqn (38c), u_1^{p} is defined by an integration. Likewise, eqn (38b) defines $\ddot{p}$. It is evident that these solutions should satisfy the first eqn (38a) too.

To be more specific, eliminating $\ddot{p}$ from eqns (38a, b) and $\ddot{u}_1$ using the incompressibility condition, eqn (38b), we derive the following governing equation for $\ddot{u}_2$.

$$\left(1+\frac{1}{\lambda}\right)\frac{\partial^4 \ddot{u}_2}{\partial X_2^4} + \left[(1+\gamma_2)\lambda + (1+\gamma_1)\left(1+\frac{1}{\lambda}\right)\right]\frac{\partial^4 \ddot{u}_2}{\partial X_1^2 \partial X_2^2}$$
$$+\lambda\frac{\partial^4 \ddot{u}_2}{\partial X_1^4} = -\frac{\partial^2 \ddot{B}_1}{\partial X_1 \partial X_2} + \lambda\frac{\partial^2 \ddot{B}_2}{\partial X_1^2} + \frac{1}{\lambda\lambda_1}\frac{\partial^3 \ddot{B}_3}{\partial X_2^3} + \frac{1+\gamma_1}{\lambda\lambda_1}\frac{\partial^3 \ddot{B}_3}{\partial^2 X_1 \partial X_2}. \quad (45)$$

Since a solution for $\ddot{u}_2$ of eqn (45) can be found, the incompressibility condition, eqn (40), yields $\ddot{u}_1$ by:

$$\ddot{u}_1 = \int \frac{1}{\lambda\lambda_1}\left[\ddot{B}_3 - \lambda_1 \frac{\partial \ddot{u}_2}{\partial X_2}\right]\mathrm{d}X_1 + uu_1(X_2). \quad (46)$$

Likewise, from eqn (38b) the pressure $\ddot{p}$ is defined by,

$$\ddot{p} = \frac{1}{\lambda_1}\int\left[-\ddot{B}_2 + (1+a_2)\frac{\partial^2 \ddot{u}_2}{\partial X_2^2} + \frac{\partial^2 \ddot{u}_2}{\partial X_1^2}\right]\mathrm{d}X_2 + pp(X_1). \quad (47)$$

The functions $uu_1(X_2)$ and $pp(X_1)$ have to satisfy eqn (40) too. As soon as the partial solution has been located, the solution for the second order displacement field, eqn (45), will be completely defined by the boundary conditions.

It is evident that further general analysis of the problem is not feasible. Therefore, the details that are many and important will be discussed in the application that is studied in the next section.

5. APPLICATION

The proposed theory is applied to a thick plate of Mooney–Rivlin material with nondimensionalized strain energy density of the type:

$$W = 8(I_1 - 3) + (I_2 - 3). \quad (48)$$

The coefficients are compatible with the limits given by Treloar (1958). The plate is of unit length $l_1 = 1$, while l_2 is taken to equal 0.30. Looking for solutions of the mode $\Omega = \pi/2$, the critical stretch $\lambda_{1,\mathrm{cr}}$ will be defined by the eqn (37b). It is found that in the present case,

$$\lambda_{1,\mathrm{cr}} = 0.925 \quad \text{and} \quad \lambda_{\mathrm{cr}} = 1.167. \quad (49)$$

The coefficient M in eqn (37a) is equal to

$$M = -1.198. \tag{50}$$

Furthermore,

$$\mathring{\sigma}_{11} = \mathring{T}/0.6 = -1.21325. \tag{51}$$

The last of eqn (36) gives:

$$\Omega_1 = \pi/2 \quad \text{and} \quad \Omega_2 = 0.838\pi. \tag{52}$$

Furthermore,

$$U_2(X_2) = -1.198 \cosh(\pi X_2/2) + \cosh(1.168\pi X_2/2) \tag{53}$$

$$U_1(X_2) = -\frac{1.198}{1.168}\sinh(\pi X_2/2) + \sinh(1.168\pi X_2/2) \tag{54}$$

$$P(X_2) = 0.463\pi(1.168^2 - 1)\sinh(1.168\pi X_2/2). \tag{55}$$

Hence, eqns (36a–c) present the first order displacement solution by:

$$\dot{u}_1 = -\xi \sin(\pi X_1/2) U_1(X_2) \tag{56a}$$

$$\dot{u}_2 = \xi \cos(\pi X_1/2) U_2(X_2) \tag{56b}$$

$$\dot{P} = \xi \cos(\pi X_1/2) P(X_2). \tag{56c}$$

Substituting for the first order solution into the second order governing equations, eqns (38a–c), we get, with the help of the Mathematica computerized pack,

$$\ddot{B}_1(\dot{u}_1, \dot{u}_2, \dot{P}) = -\pi^3\xi^2[-1.34 \sin(\pi X_1) + 1.36 \sin(\pi X_1)\cosh(0.08\pi X_2) + 0.015 \sin(\pi X_1)\cosh(1.08\pi X_2)]. \tag{57a}$$

$$\ddot{B}_2(\dot{u}_1, \dot{u}_2, \dot{P}) = -\pi^3\xi^2[0.007 \sinh(0.8\pi X_2) - 0.033 \cos(\pi X_1)\sinh(0.8\pi X_2) - 0.71 \sinh(\pi X_2) + 1.43 \sinh(1.08\pi X_2)] \tag{57b}$$

$$\begin{aligned} \ddot{B}_3(\dot{u}_1, \dot{u}_2, \dot{P}) = -\xi^2\pi^2/2[&-0.61 \sin^2(\pi X_1/2)\cosh^2(\pi X_2/2) \\ &+ 1.21 \cosh(\pi X_2/2)\cosh(0.58\pi X_2)\sin^2(\pi X_1/2) \\ &- 0.58 \cosh^2(0.58\pi X_2)\sin^2(\pi X_1/2) \\ &- 0.61 \cos^2(\pi X_1/2)\sinh^2(\pi X_2/2) \\ &+ 1.20 \cos^2(\pi X_1/2)\sinh(\pi X_2/2)\sinh(0.58\pi X_2) \\ &- 0.58 \cos^2(\pi X_1/2)\sinh^2(0.58\pi X_2)]. \end{aligned} \tag{57c}$$

In addition a resultant equation similar to eqn (44) but for $\ddot{u}_1$ becomes:

$$4.91\frac{\partial^4 \ddot{u}_1}{\partial^4 X_1} + 8.51\frac{\partial^4 \ddot{u}_1}{\partial^2 X_1\,\partial^2 X_2} + 3.6\frac{\partial^4 \ddot{u}_1}{\partial^4 X_2} = -\xi^2[416.45 \sin(3.14X_1) - 413.15 \cosh(0.26X_2)\sin(3.14X_1) + 0.65 \cosh(3.40X_2)\sin(3.14X_1)], \tag{58}$$

The general solution to eqn (58) preserving the symmetry of problem consists only of the particular solution. That is evident since the linear homogeneous solution to eqn (58) is antisymmetric which does not conform with the physics of the second order solution requiring only symmetric behavior.

$$\ddot{u}_1 = \xi^2[-0.87\sin(\pi X_1)+0.87\cosh(0.26X_2)\sin(\pi X_1) \\ +0.056\cosh(3.40X_2)\sin(\pi X_1)+a_{30}X_1^3+a_{12}X_1X_2^2+a_{10}X_1 \\ +\sin(\pi X_1)[-0.856a_1\cosh(\pi X_2)+a_2\cosh(1.167\pi X_2)]]. \quad (59)$$

Substituting for $\ddot{u}_1$ into the incompressibility condition and integrating, the solution for $\ddot{u}_2$ is derived by:

$$\ddot{u}_2 = \xi^2[-1.1677a_{10}X_2-3.50a_{30}X_1^2X_2-0.389a_{12}X_2^3 \\ +0.073\sinh(0.26X_2)+12.19\cos(\pi X_1)\sinh(0.26\,X_2) \\ -12.18\cos(\pi X_1)\sinh(0.26\,X_2)+0.52\sinh(3.14X_2) \\ -0.94\sinh(3.41X_2)+0.0056\cos(3.14X_1)\sinh(3.41X_2) \\ -0.06\cos(3.14X_1)\sinh(3.41X_2)+0.42\sinh(3.67X_2) \\ +uu_2(X_1)+\cos(3.14X_1)[a_1\sinh(3.14X_2) \\ +a_2\sinh(3.41X_2)]]. \quad (60)$$

Likewise, eqn (38b) yields:

$$\ddot{P} = 2/(W_1+W_2)\ddot{p} = \xi^2[0.66+4.54a_{12}X_2^2+13.63a_{30}X_2^2 \\ +8.013\cos(3.14X_1)-0.75\cosh(0.26X_2) \\ +1754.06\cos(3.14X_1)\cosh(0.26X_2) \\ -1762.25\cos(3.14X_1)\cosh(0.26X_2) \\ -6.36\cosh(3.14X_2)+13.32\cosh(3.41X_2) \\ +0.053\cos(3.14X_1)\cosh((3.41X_2) \\ +0.119\cos(3.14X_1)\cosh(3.41X_2) \\ -6.86\cosh(3.66X_2)+pp(X_1)-3.89X_2\frac{\mathrm{d}(uu_2(X_1))}{\mathrm{d}(X_1)} \\ +1.21a_2\pi\cos(3.14X_1)\cosh(3.41X_2)]. \quad (61)$$

It is apparent that the solution (58)–(60) of the system (40a–c) depends on the parameters $\{a_1, a_2, a_{10}, a_{12}, a_{30}\}$ and the unknown functions $pp(X_1)$ and $uu_2(X_1)$. Nevertheless, substituting for the solution into the first governing eqn (40a) we get the relation:

$$\frac{\mathrm{d}[pp(X_1)]}{\mathrm{d}[X_1]} = 0.925(2a_{12}+6a_{30})X_1+3.89X_2\frac{\mathrm{d}^3[uu_2(X_1)]}{\mathrm{d}^3[X_1]}. \quad (62)$$

It is evident that $uu_2(X_1)$ is at most a quadratic function of X_1 and,

$$pp(X_1) = 0.925[a_{12}+3a_{30}]X_1^2+pc. \quad (63)$$

The various parameters will be defined by the boundary conditions (40d,e). Indeed, with the help of the Mathematica pack we get:

$$\ddot{\sigma}_{21}(X_1, \pm 0.3) = \pm[2.16a_{12}X_1 - 8.84a_{30}X_1 - 24.91a_1 \sin(\pi X_1) - 35.30a_2 \sin(\pi X_1) + 1.91 \sin(\pi X_1)]\xi^2 + 4.2\frac{d[uu_2(X_1)]}{dX}\xi^2 = 0 \quad (64)$$

and

$$\ddot{\sigma}_{22}(X_1, \pm 0.3) = \Big[-8.41a_{10} - 1.135a_{12} - 1.135a_{30} - 0.019 - 0.925pc - 0.856a_{12}X_1^2 - 27.79a_{30}X_1^2 + 33.43a_1 \cos(3.14X_1) + 38.21a_2 \cos(3.14X_1) - 2.285\cos(3.14X_1) + 1.08\frac{d[uu_2^2(X_1)]}{dX_1^2}\Big]\xi^2 = 0 \quad (65)$$

$$\ddot{T} = \int_{-0.3}^{0.3} \ddot{\sigma}_{11}\, dX_1 = [5.11a_{10} + 15.32a_{30} + 15.86a_1 + 22.47a_2 - 0.90 - 0.65pc - 0.6a_{12}X_1^2 - 1.8a_{30}X_1^2]\xi^2 \quad (66)$$

$$\ddot{u}_2(\pm 1, 0) = \xi^2 uu_2(1) = 0 \quad (67)$$

Since (64) should be identically zero, the following relations have to be satisfied.

$$2.16a_{12} - 8.836a_{30} = 0 \quad (68)$$

$$-24.9a_1 + 35.30a_2 + 1.91 = 0 \quad (69)$$

$$uu_2(X_1) = \text{constant}. \quad (70)$$

Likewise, eqn (65) yields

$$-8.41a_{10} - 1.14a_{12} - 1.13a_{30} - 0.019 - 0.925pc = 0 \quad (71)$$

$$-0.8564a_{12} - 27.79a_{30} = 0 \quad (72)$$

$$33.43a_1 + 38.21a_2 - 2.29 = 0. \quad (73)$$

In addition eqn (66) is reduced to:

$$\ddot{T} = (2.55a_{10} + 7.65a_{30} + 7.93a_1 + 11.24a_2 - 0.45 - 0.32pc)\xi_1^2. \quad (74)$$

Furthermore, eqns (67) and (70) state that

$$uu_2(X_1) = 0. \quad (75)$$

Equations (68)–(73) uniquely define the various parameters of the solution. Indeed, eqns (68) and (72) dictate that

$$a_{12} = a_{30} = 0. \quad (76)$$

Furthermore, eqns (68) and (72) define the parameters as follows:

$$a_1 = 0.072171, \quad a_2 = -0.003167. \quad (77)$$

Finally, eqns (70) and (73) give the values of the parameters a_{10} and pc:

$$a_{10} = 0.092343 + 0.0916\ddot{T}/\xi_1^2 \quad (78)$$

$$pc = -0.8601\xi^2 - 0,8326\ddot{T}/\xi_1^2 \quad (79)$$

with $\ddot{\mathrm{T}} < 0$.

The second order displacement solution depends on the parameter ξ_1 as the first order does. The parameter ξ_1 will be defined by the third order problem using bifurcation methods. Apart from the unknown parameter ξ_1, up to this stage, the solution has uniquely been defined. Indeed substituting the parameters, the final expression of the solution is:

$$\begin{aligned}\ddot{u}_1 = \xi^2[&-0.87\sin(\pi X_1) + 0.87\cosh(0.26X_2)\sin(\pi X_1) \\ &+ 0.056\cosh(3.40X_2)\sin(\pi X_1) + 0.09234X_1 \\ &+ \sin(\pi X_1)[-0.856\,0.072\cosh(\pi X_2) - 0.03167\cosh(1.167\pi X_2)]] + 0.0916\ddot{T} \quad (80)\end{aligned}$$

$$\begin{aligned}\ddot{u}_2 = {} & -0.1070\ddot{T} + \xi^2[-0.10783X_2 \\ & + 0.073\sinh(0.26X_2) + 12.19\cos(\pi X_1)\sinh(0.26X_2) \\ & - 12.18\cos(\pi X_1)\sinh(0.26X_2) + 0.52\sinh(3.14X_2) \\ & - 0.94\sinh(3.41X_2) + 0.0056\cos(3.14X_1)\sinh(3.41X_2) \\ & - 0.06\cos(3.14X_1)\sinh(3.41X_2) + 0.42\sinh(3.67X_2) \\ & + \cos(3.14X_1)[0.07221\sinh(3.14X_2) \\ & - 0.003167\sinh(3.41X_2)]]. \quad (81)\end{aligned}$$

Likewise, eqn (60) yields:

$$\begin{aligned}\ddot{P} = 2/(W_1 + W_2)\ddot{p} = {} & -0.8326\ddot{T} + \xi^2[-0.20 + \\ & + 8.013\cos(3.14X_1) - 0.75\cosh(0.26X_2) \\ & + 1754.06\cos(3.14X_1)\cosh(0.26X_2) \\ & - 1762.25\cos(3.14X_1)\cosh(0.26X_2) \\ & - 6.36\cosh(3.14X_2) + 13.32\cosh(3.41X_2) \\ & + 0.053\cos(3.14X_1)\cosh(3.41X_2) \\ & + 0.119\cos(3.14X_1)\cosh(3.41X_2) \\ & - 6.86\cosh(3.66X_2) \\ & - 0.012\cos(3.14X_1)\cosh(3.41X_2)]. \quad (82)\end{aligned}$$

6. CONCLUSION

Standard bifurcation methods were applied for the definition of the post-critical equilibrium path of a thick highly elastic incompressible plate under uniaxial thrust. Second order barreling deformation was revealed for the first order flexure mode. That incremental deformation was described as a mono parametric family of functions. The second order displacement vector and the first order as well are expected to be defined from the study of the third order linearized governing equilibrium problem. Since the calculus of the method is too extensive, applications are possible only through computerized algebra packs, such as Mathematica. Using the present method, various experimental results (Beatty and Hook, 1968; Beatty and Dadras, 1976) have a reasonable explanation. We suspect that the experimental evidence of the barreling behavior should not be attributed to the transition from the flexure to the barreling first order modes, as it is underlined in Beatty and Hook

(1968), and Beatty and Dadras (1976), but in the existence of the second order barreling mode escorting the first order flexural mode.

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